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The Recovery of Distorted Band-Limited Signals

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In this paper we devise a method for recovering band limited signals which have been subjected to a distorting procedure called companding.

A band limited signal is a function in L^2 whose Fourier transform vanishes outside of a bounded interval. In practice most real signals are in this class since they have no indefinitely high frequency components. The operation of companding a signal $f(t)$ consists of replacing it by $\varphi[f(t)]$, where $\varphi(x)$ is a given function. This operation does not in general preserve the band-limitedness of $f(t)$.

We reproduce an unpublished uniqueness theorem of *A.* Beurling which shows that the knowledge of the transform of the companded signal on the interval where the transform of the original signal does not vanish determines uniquely the original signal, provided the companding function is monotonic increasing.

We then prove the following existence theorem. To each function in L^2 defined only on a bounded interval there is one and only one band limited function with band contained in this interval, the Fourier transform of whose companded version coincides with the given function.

Our method is constructive and proceeds via a stable iteration scheme (Picard iterations). The convergence of our method requires that the companding function have a derivative which is bounded away from zero and infinity.

I. INTRODUCTION

In this paper we will present an existence and uniqueness proof for the solution of a nonlinear functional equation.[1] This equation arises

[1] The work discussed in this paper was done during the summer of 1958 while the authors were employed at the Bell Telephone Laboratories. This paper is a revision of a report by the authors issued by the Bell Telephone Laboratories [1].

as a model of a signal recovery problem in which so-called band limited functions are distorted by a procedure termed companding and subsequently band-limiting. The companding operation is nonlinear and does not preserve the band-limitedness of the signal. Thus the distortion apparently involves a loss of information.

A signal is a function $f(t)$ which is square integrable. In communications we often deal with signals which are presumed to have no very high frequency components. This assumption has been shown to be reasonable by observation. In addition, since many signal creating mechanisms, like the human voice, are composed of mechanical devices, it is pausible to view them as responding only with a bounded set of harmonics. Even electronic signal handling mechanisms are felt to filter out sufficiently large frequency components.

We make this class of signals precise by use of Fourier transforms and say that a signal $f(t)$ is band limited with band $(-\Omega, \Omega)$ if its Fourier transform $F(\omega)$ (denoted operationally by Tf) vanishes for $|\omega| > \Omega$.

To compand $f(t)$ is to transform it into $\varphi[f(t)]$ where $\varphi(x)$ is a monotonic function. Companding may be viewed as nonlinear amplification. However, it has a more specific use in improving the performance of transmission systems. Since transmission systems do not respond very well to very high or very low signal levels, the typical companding function in this application has a very high slope near zero and tapers off rapidly ar $\pm\infty$. Thus the weaker parts of the signal are amplified and the stronger parts deemphasized preliminary to transmission.

We assume henceforth that the companding operation takes a square integrable function into a square integrable function. This can be assured by such requirements as $\varphi(0) = 0$, and $\varphi' < M$. However, it is generally true that $\varphi[f(t)]$ will not be band limited even when $f(t)$ is.

Now we suppose that the companded signal $\varphi[f(t)]$ is handled by a band-limiting device with band $(-\Omega, \Omega)$. For example, the companded signal may be sent down a channel which does not pass frequency components larger in magnitude than Ω. The effect of such an operation is to produce a signal whose Fourier transform $S(\omega)$ agrees with $T\varphi[f]$ on the band $(-\Omega, \Omega)$ but vanishes outside of this band. Thus if

$$\chi(\omega) = \begin{cases} 1, & |\omega| \leqslant \Omega \\ 0, & \text{otherwise}, \end{cases} \tag{1.1}$$

we have

$$S(\omega) = \chi(\omega) T\varphi[f(t)]. \tag{1.2}$$

The problem presented in this paper is to determine $f(t)$ from (1.2) when $S(\omega)$ is given.

We first reproduce a uniqueness theorem of A. Beurling which states that at most one band limited solution with band $(-\Omega, \Omega)$ of (1.2) exists. That is, the knowledge of the Fourier transform of a companded band-limited signal only on the domain where the Fourier transform of the original signal does not vanish (the band of our signals) is sufficient to determine the signal uniquely. This theorem is a consequence of the Plancherel theorem, and requires only that $\varphi(x)$ be monotone increasing.

Under the additional hypothesis that $\varphi'(x)$ exist and be bounded above and below by positive constants, we produce an existence theorem: *To each square integrable function $S(\omega)$ there exists one (and only one) band-limited signal $f(t)$ with band $(-\Omega, \Omega)$, which satisfies* (1.2). Our existence theorem proceeds by the method of successive approximations (Picard iterations), and thus is constructive.

We conclude the paper with some remarks concerning the stability of the iteration scheme and convergence in the L^∞ norm. In the following section we state and prove our results.

II. Statement and Proofs

DEFINITION: *A signal is a function $f(t) \in L^2$ $(-\infty, \infty)$, i.e.,*

$$||f||^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty \tag{2.1}$$

DEFINITION: *A signal is band limited with band $(-\Omega, \Omega)$ if its Fourier transform*

$$Tf(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \, e^{-i\omega t} \, dt \tag{2.2}$$

vanishes for $|\omega| > \Omega$.

LEMMA 1: *The set of signals with band $(-\Omega, \Omega)$ is a closed subspace of L^2.*

PROOF: Suppose a sequence of signals with band $(-\Omega, \Omega)$, f_n, converges in the mean square to f. Clearly $f \in L^2$. Then by Plancherel's theorem the corresponding Fourier transform converge likewise, viz.,

$$\int_{-\infty}^{\infty} |f_n(t) - f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F_n(\omega) - F(\omega)|^2 \, d\omega. \tag{2.3}$$

Thus $F(\omega)$ must vanish for $|\omega| > \Omega$ and $f(t)$ has band $(-\Omega, \Omega)$.

REMARK: We fix Ω once and for all, and denote by B the subspace described in the previous lemma. In addition we denote by $\chi(\omega)$, the characteristic function of the band, i.e.

$$\chi(\omega) = \begin{cases} 1, & |\omega| \leqslant \Omega \\ 0, & \text{otherwise} \end{cases}$$

DEFINITION: *A compandor is a monotonic function $\varphi(x)$ which has the property that $\varphi[f(t)] \in L^2$ if $f(t) \in L^2$.*

We now state and prove the

UNIQUENESS THEOREM:[2] *The functional equation*

$$S(\omega) = \chi(\omega) T\varphi[f(t)] \tag{2.4}$$

where $S(\omega) \in L^2$ and $\varphi(x)$ is a monotonic increasing compandor has at most one solution $f(t) \in B$.

PROOF: Suppose $f_1(t)$ and $f_2(t)$ are solutions of (2.4) which lie in B. Then by Plancherel's theorem we have

$$\int_{-\infty}^{\infty} [f_1(t) - f_2(t)]\, [\varphi(f_1(t)) - \varphi(f_2(t))]\, dt$$

$$= \int_{-\infty}^{\infty} [F_1(\omega) - F_2(\omega)]\, \overline{[T\varphi(f_1(t)) - T\varphi(f_2(t))]}\, d\omega. \tag{2.5}$$

The first factor in the integrand of the right-hand member of(2.5) vanishes for $|\omega| > \Omega$, since f_1 and $f_2 \in B$. The second factor vanishes for $|\omega| \leqslant \Omega$, since for these values of ω both elements of this factor are equal to $S(\omega)$ by hypothesis. Thus

$$\int_{-\infty}^{\infty} [f_1(t) - f_2(t)]\, [\varphi(f_1(t)) - \varphi(f_2(t))]\, dt = 0. \tag{2.6}$$

From this the monotony of φ implies that $f_1 \equiv f_2$ and thus the theorem.

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[2] This theorem, which was reported to us in a private communication, is due to A. Beurling.

We now state and prove the

EXISTENCE THEOREM: *If the compandor $\varphi(x)$ is differentiable and m and M are positive constants such that*

$$m \leqslant \varphi' \leqslant M,$$

then the functional equation (2.4) *has a solution $f(t) \in B$ for each $S(\omega) \in L^2$.*

PROOF: The theorem will be proved by constructing a sequence of successive approximations f_n to f. The necessary iteration formula is derived from the equation

$$f = cT^{-1}S(\omega) + T^{-1}\chi T(f - c\varphi[f]), \tag{2.7}$$

where c is a constant.

We remark that a solution of (2.7) in B is a solution of (2.4) in B. This is seen to be the case as follows: Since $T^{-1}\chi T$ is the identity map in B, (2.7) reduces to

$$0 = cT^{-1}S(\omega) - T^{-1}\chi Tc\varphi[f]. \tag{2.8}$$

Since T^{-1} is an isomorphism of L^2 itself and $T^{-1}\chi T$ is a linear operator, a solution of (2.8) is a solution of

$$S(\omega) = \chi T\varphi[f], \tag{2.9}$$

i.e., of (2.4) and conversely. The arbitrary constant c is to be specified in a manner depending upon m and M, and controls the rate of convergence of our iterations. The advantage of considering (2.7) instead of (2.4) is that it is a map of the second kind and thus naturally induces a procedure of successive approximations. A second advantage is that it is a map of L^2 into B, a fortiori of B into B. This latter property is a prerequisite for applying the method of successive approximations.

We now claim that the sequence of functions $f_0, f_1, \ldots$, where

$$f_{n+1} = cT^{-1}S(\omega) + T^{-1}\chi T(f_n - c\varphi[f_n]) \tag{2.10}$$

converges to a solution in B of (2.10), provided $f_0 \in B$ and c is appropriately chosen.

If the f_n form a Cauchy sequence they will converge to a function $f \in L^2$, since L^2 is a complete space. Also, since each $f_n \in B$ and B is closed (Lemma 1), f will be in B. To show that the f_n form a Cauchy sequence we proceed as follows:

$$\begin{aligned} ||f_{n+1} - f_n||^2 &= ||T^{-1}\chi T[f_n - f_{n-1} - c(\varphi(f_n) - \varphi(f_{n-1}))]||^2 \\ &= ||\chi T[f_n - f_{n-1} - c(\varphi(f_n) - \varphi(f_{n-1}))]||^2, \end{aligned} \tag{2.11}$$

since the Fourier transform is an isometry of L^2,

$$= \int_{-\infty}^{\infty} |\chi(\omega) T[f_n - f_{n-1} - c(\varphi(f_n) - \varphi(f_{n-1}))]|^2 \, d\omega$$

$$= \int_{-\Omega}^{\Omega} |T[f_n - f_{n-1} - c(\varphi(f_n) - \varphi(f_{n-1}))]|^2 \, d\omega$$

$$\leqslant ||T[f_n - f_{n-1} - c(\varphi(f_n) - \varphi(f_{n-1})]||^2$$

$$= ||f_n - f_{n-1} - c(\varphi(f_n) - \varphi(f_{n-1}))||^2$$

$$= \int_{-\infty}^{\infty} |f_n - f_{n-1}|^2 \left| 1 - c \, \frac{\varphi(f_n) - \varphi(f_{n-1})}{f_n - f_{n-1}} \right|^2 dt$$

$$\leqslant \max_x |1 - c\varphi'(x)|^2 \, ||f_n - f_{n-1}||^2,$$

by the mean value theorem.

This shows that the f_n form a Cauchy sequence if

$$\theta^2 = \max_x |1 - c\varphi'(x)|^2 < 1. \text{ [3]} \tag{2.12}$$

Since $0 < m \leqslant \varphi' \leqslant M$, this can be arranged by choosing c to be any positive constant less than $2/M$. The estimate (2.11) also shows that the equation (2.9) is continuous in f and thus we may take the limit as $n \to \infty$ formally in (2.10). Thus the limit f of the f_n is the sought for solution and the theorem is proved.

The convergence of the f_n to f as demonstrated in the above theorem is in the mean square sense, i.e.

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} |f_n - f|^2 \, dt = 0. \tag{2.13}$$

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[3] This is the case since

$$||f_{n+p} - f_n|| = || \sum_{j=n}^{n+p-1} f_{j+1} - f_j ||$$

$$\leqslant \sum_{j=n}^{n+p-1} ||f_{j+1} - f_j|| \leqslant \sum_{j=n}^{n+p-1} \theta^j \, ||f_1 - f_0|| \leqslant \frac{\theta^n}{1 - \theta} \, ||f_1 - f_0||.$$

It is also true that f_n converges to f in the maximum norm, i.e.

$$\lim_{n\to\infty} \max_t |f_n(t) - f(t)| = 0. \tag{2.14}$$

This fact is implied by the following

LEMMA 2:[4] *If $f \in B$ then*

$$\max_t |f(t)| < \sqrt{\frac{\Omega}{\pi}}\, ||f||. \tag{2.15}$$

PROOF:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t}\, Tf(t)\, d\omega. \tag{2.16}$$

Since $f \in B$, $Tf \equiv 0$ for $|\omega| > \Omega$, so that

$$\max_t |f(t)| \leqslant \frac{1}{\sqrt{2\pi}} \max_t \left| \int_{-\Omega}^{\Omega} e^{i\omega t}\, Tf(t)\, d\omega \right|. \tag{2.17}$$

Then by Schwarz' inequality

$$\max_t |f(t)| \leqslant \frac{1}{\sqrt{2\pi}} \left[\int_{-\Omega}^{\Omega} d\omega \int_{-\infty}^{\infty} |Tf(t)|^2\, d\omega \right]^{1/2} = \sqrt{\frac{\Omega}{\pi}}\, ||f||. \tag{2.18}$$

REMARK 1: The method of successive approximations used in the proof of the existence theorem proceeds essentially by demonstrating that the sequence $f - f_n$ is bounded by a geometric series. This series for the present case is

$$\theta^n\, ||f - f_0||. \tag{2.19}$$

Thus an error ΔS, in $S(\omega)$, would produce an error in f bounded by

$$\frac{c}{1-\theta}\, ||\Delta S||. \tag{2.20}$$

This shows the stability of the signal recovery process.

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[4] We are indebted to H. Pollak for this observation.

REMARK 2: The rate of convergence of the iterates as well as the degree of stability is governed by the size of θ. The closer θ is to zero the faster the iterates will converge and the less sensitive will the limit be to errors in $S(\omega)$. From (2.12) we see that the constant c can be chosen so as to adjust the size of θ to achieve these desirable effects.

REMARK 3: A circuit on an analogue computer has been devised by one of the authors to perform numerical calculations based on this iteration scheme. His studies of this matter, as well as a detailed analysis of the stability property appear in [2].

REFERENCES

1. LANDAU, H. J. AND MIRANKER, W. L. A method for solving functional equations and its application to a signal recovery problem. *Bell Telephone Lab. Tech. Mem.*, November 6, 1958.
2. LANDAU, H. J. On the recovery of a band-limited signal, after instantaneous companding and subsequent band-limiting. *Bell System Tech. J.*, March, 1960.